

The boson-boson bound state in the massive Schwinger model

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Abstract

We use (fermion) mass perturbation theory for the massive Schwinger model to compute the boson-boson bound state mass in lowest order. For small fermion mass the lowest possible Fock state turns out to give the main contribution and leads to a second order result for the bound state mass.

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1 Introduction

The massless Schwinger model - which is two-dimensional QED with one massless fermion - is wellknown to be exactly soluble ([1] - [4], [8] - [10]). The spectrum of the model consists of one free, massive boson with Schwinger mass $\mu_0^2 = \frac{e^2}{\pi}$, which may be interpreted as a fermion-antifermion bound state ([5], [6]). Besides, instantons and a nontrivial vacuum structure (θ -vacuum) are present, and a fermion condensate is formed ([7] - [12]).

The massive Schwinger model (with one massive fermion) is no longer exactly soluble, and the physical particle (the Schwinger boson) is no longer free ([13] - [16]). However, the nontrivial features of the massless model (instantons, θ -vacuum, fermion condensate) persist to be present in the massive case ([14] - [18]).

The known exact solution of the massless model can be used for a mass perturbation expansion of the massive theory that preserves all the nontrivial features of the model ([19]). For the fermion condensate and Schwinger mass this was done in [17], [18].

Here we focus on the boson-boson interaction. It is mediated by an attractive force and therefore a boson-boson bound state is formed. We compute the mass of this bound state within mass perturbation theory in leading order.

2 Exact n -point functions of the massless model

The vacuum functional (and Green functions) of the massive Schwinger model may be inferred from n -point functions of the massless Schwinger model via an expansion in the fermion mass. Indeed, we may write for the Euclidean vacuum functional ($k \dots$ instanton number)

$$Z(m, \theta) = \sum_{k=-\infty}^{\infty} e^{ik\theta} Z_k(m) \quad (1)$$

where

$$\begin{aligned} Z_k(m) &= N \int D\bar{\Psi} D\Psi D A_k^\mu e^{\int dx \left[\bar{\Psi}(i\partial - eA_k + m)\Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]} \\ &= N \int D\bar{\Psi} D\Psi D\beta_k \sum_{n=0}^{\infty} \frac{m^n}{n!} \prod_{i=1}^n \int dx_i \bar{\Psi}(x_i) \Psi(x_i) \cdot \\ &\quad \cdot \exp\left\{ \int dx \left[\bar{\Psi}(i\partial - \epsilon_{\mu\nu} \gamma^\mu \partial^\nu \beta_k) \Psi + \frac{1}{2e^2} \beta_k \square^2 \beta_k \right] \right\} \end{aligned} \quad (2)$$

($eA_\mu = \epsilon_{\mu\nu} \partial^\nu \beta$ corresponding to Lorentz gauge). Therefore, the perturbative computation of $Z(m, \theta)$ is traced back to the computation of scalar VEVs ($\langle \prod_i S(x_i) \rangle_0$, $S(x) \equiv \bar{\Psi}(x) \Psi(x)$) for the massless Schwinger model and some space time integrations. It is useful to rewrite the scalar densities in terms of chiral ones, $S(x) = S_+(x) + S_-(x)$, $S_\pm \equiv \bar{\Psi} P_\pm \Psi$, because in this case only a definite instanton sector $k = n_+ - n_-$ contributes to the VEV $\langle \prod_{i=1}^{n_+} S_+(x_i) \prod_{j=1}^{n_-} S_-(x_j) \hat{O} \rangle_0$, where \hat{O} is a chirality neutral operator (e.g. a product of vector currents). A general chiral VEV may be computed exactly (see e.g. [7]–[9], [20], [17]),

$$\langle S_{H_1}(x_1) \cdots S_{H_n}(x_n) \rangle_0 = \left(\frac{\Sigma}{2} \right)^n \exp \left[\sum_{i < j} (-)^{\sigma_i \sigma_j} 4\pi D_{\mu_0}(x_i - x_j) \right] \quad (3)$$

where $\sigma_i = \pm 1$ for $H_i = \pm$, D_{μ_0} is the massive scalar propagator,

$$D_{\mu_0}(x) = -\frac{1}{2\pi} K_0(\mu_0|x|), \quad \tilde{D}_{\mu_0}(p) = \frac{-1}{p^2 + \mu_0^2}, \quad (4)$$

($K_0 \dots$ McDonald function) and Σ is the fermion condensate of the massless Schwinger model,

$$\Sigma = \langle \bar{\Psi} \Psi \rangle_0 = \frac{e^\gamma}{2\pi} \mu_0 \quad (5)$$

($\gamma \dots$ Euler constant). The index 0 for μ_0 indicates that it is the order zero result, the index 0 for the VEVs means that they are computed with respect to the massless Schwinger model. From this $Z(m, \theta)$ may be computed (see [17] for details, [21] for its physical implications),

$$Z(m, \theta) = e^{V\alpha(m, \theta)},$$

$$\alpha(m, \theta) = \mu_0^2 \left[\frac{m}{\mu_0} \frac{\Sigma}{2\mu_0} 2 \cos \theta + \frac{m^2}{\mu_0^2} \left(\frac{\Sigma}{2\mu_0} \right)^2 (E_+ \cos 2\theta + E_-) + o\left(\frac{m^3}{\mu_0^3}\right) \right] \quad (6)$$

($V \dots$ space time volume) where E_+ and E_- are numbers ($E_+ = -8.9139$, $E_- = 9.7384$).

In order to compute VEVs for the massive Schwinger model one has to insert the corresponding operators into the path integral (1), (2) and divide by the vacuum functional $Z(m, \theta)$:

$$\langle \hat{O} \rangle_m = \frac{1}{Z(m, \theta)} \langle \hat{O} \sum_{n=0}^{\infty} \frac{m^n}{n!} \prod_{i=1}^n \int dx_i \bar{\Psi}(x_i) \Psi(x_i) \rangle_0 \quad (7)$$

Via the normalization all volume factors cancel completely, as it certainly has to be.

It is wellknown that the Schwinger boson ϕ is related to the vector current, $J_\mu \sim \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \phi$, therefore, for its study vector current correlators are needed. E.g. for the propagator $\langle \phi(y_1) \phi(y_2) \rangle_m$ and for the computation of the Schwinger mass one needs the n -point functions

$$\begin{aligned} \langle \phi(y_1) \phi(y_2) \prod_{i=1}^n S_{H_i}(x_i) \rangle_0 &= \left(\frac{\Sigma}{2} \right)^n e^{4\pi \sum_{k<l}^n (-)^{\sigma_k \sigma_l} D_{\mu_0}(x_k - x_l)} \\ &\cdot \left[D_{\mu_0}(y_1 - y_2) + 4\pi \left(\sum_{i=1}^n (-)^{\sigma_i} D_{\mu_0}(x_i - y_2) \right) \left(\sum_{j=1}^n (-)^{\sigma_j} D_{\mu_0}(x_j - y_1) \right) \right]. \end{aligned} \quad (8)$$

For the Schwinger mass one finds the explicit result ([18])

$$\frac{\mu_2^2}{\mu_0^2} = 1 + 8\pi \frac{m}{\mu_0} \frac{\Sigma}{2\mu_0} \cos \theta + 16\pi^2 \frac{m^2}{\mu_0^2} \left(\frac{\Sigma}{2\mu_0} \right)^2 (A \cos 2\theta + B) + o\left(\frac{m^3}{\mu_0^3}\right) \quad (9)$$

where

$$A = -0.6599, \quad B = 1.7277.$$

In order to study the boson-boson interaction we need the four-point function $\langle \phi(y_1) \phi(y_2) \phi(y_3) \phi(y_4) \rangle_m$ and therefore the n -point functions

$$\langle \phi(y_1) \phi(y_2) \phi(y_3) \phi(y_4) \prod_{i=1}^n S_{H_i}(x_i) \rangle_0 = \left(\frac{\Sigma}{2} \right)^n e^{4\pi \sum_{k<l}^n (-)^{\sigma_k \sigma_l} D_{\mu_0}(x_k - x_l)}.$$

$$\begin{aligned}
& \cdot \left[D_{\mu_0}(y_1 - y_2) D_{\mu_0}(y_3 - y_4) + \text{perm.} + \right. \\
& 4\pi D_{\mu_0}(y_1 - y_2) \left(\sum_{i=1}^n (-)^{\sigma_i} D_{\mu_0}(y_3 - x_i) \right) \left(\sum_{j=1}^n (-)^{\sigma_j} D_{\mu_0}(y_4 - x_j) \right) + \text{perm.} + \\
& \left. 16\pi^2 \left(\sum_{i=1}^n (-)^{\sigma_i} D_{\mu_0}(y_1 - x_i) \right) \left(\sum_{j=1}^n (-)^{\sigma_j} D_{\mu_0}(y_2 - x_j) \right) \left(\sum_{k=1}^n (-)^{\sigma_k} D_{\mu_0}(y_3 - x_k) \right) \left(\sum_{l=1}^n (-)^{\sigma_l} D_{\mu_0}(y_4 - x_l) \right) \right]
\end{aligned} \tag{10}$$

where ”+ perm” means the sum of all *distinguishable* terms. Actually the first and second types of terms are disconnected, so only the third one will contribute.

3 Computation of the bound state mass

When evaluating (7, 10) for lowest order in m , one finds an interaction term

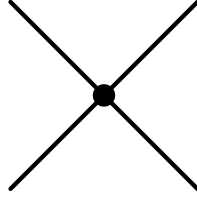


Fig. 1

with a coupling constant (the instanton sectors $k = \pm 1$ contribute, see (1))

$$16\pi^2 m \Sigma \cos \theta =: 2g, \tag{11}$$

where a convenient abbreviation was introduced.

Therefore a bound state has to be expected for $|\theta| < \frac{\pi}{2}$, where the force is attractive, and this restriction shall be assumed in the sequel.

Before continuing we want to fix the graphical notation:

$$\begin{array}{llll}
D_{\mu}(x) & \cdots & \text{—————} & \\
D_{\mu}^2(x) & \cdots & \text{~~~~~} & \text{or } \text{—————} \\
e^{+(-)4\pi D_{\mu}(x)} - 1 & \cdots & \text{~~~~~} &
\end{array}$$

Fig. 2

where the “ -1 ” in the last expression stems from the cluster expansion that is at the heart of the mass perturbation theory ([17], [18], [19]).

Actually, we want to compute the lowest order bound state mass in an approximation that resembles Φ^4 -theory, by summing a series

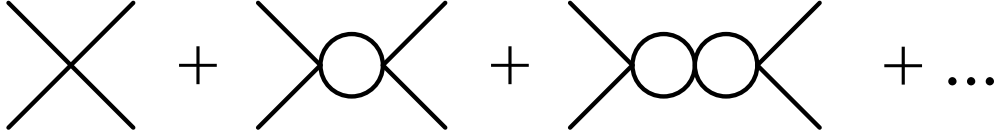


Fig. 3

At first glance, substituting the exponential $E_{\pm}(x) = \exp(\pm 4\pi D_{\mu}(x)) - 1$ by the quadratic term $\frac{16\pi^2}{2!} D_{\mu}^2(x)$ seems to be a very rough approximation. However, computing the bound state mass perturbatively in m will involve the solution of a self consistency equation for a momentum (remember Euclidean conventions!) $q^2 \sim -4\mu^2(1 - \epsilon)$, where ϵ is very small for small m . It is precisely the Fourier transform of $D_{\mu}^2(x)$ that has a threshold singularity at $q^2 = -4\mu^2$, therefore it will give the main contribution and the approximation is justified.

Next we fix (y_1, y_2) to form the incoming state and (y_3, y_4) to form the outgoing one.

Now we have to show that the graphs of the series depicted above (Fig. 3) actually occur in the mass perturbation series.

First, the θ dependence is very simple. Because external lines are joined in pairs, and because we select the quadratic part of the expanded exponential $\exp(\pm 4\pi D_{\mu}(x))$, all instanton sectors contribute identical terms to a given order n in the mass m . Further the instanton sector k gives $\binom{n}{n_-}$ terms, where $k = n - 2n_-$. So altogether we find

$$\sum_{n_-=1}^n \binom{n}{n_-} e^{i(n-2n_-)\theta} = 2^n \cos^n \theta \quad (12)$$

identical terms, which shall be computed next.

In second order we have (μ is an unspecified mass for the moment)

$$\begin{aligned} & \frac{m^2}{2!} \Sigma^2 \cos^2 \theta 16\pi^2 \int dx_1 dx_2 \prod_{i=1}^4 [D_{\mu}(y_i - x_1) + D_{\mu}(y_i - x_2)] \frac{(4\pi)^2}{2!} D_{\mu}^2(x_1 - x_2) \\ & \simeq 2g \int dx_1 dx_2 \frac{m}{2!} \Sigma \cos \theta \frac{16\pi^2}{2} D_{\mu}^2(x_1 - x_2) \cdot \\ & \quad \cdot (D_{\mu}(y_1 - x_1) D_{\mu}(y_2 - x_1) D_{\mu}(y_3 - x_2) D_{\mu}(y_4 - x_2) + (x_1 \leftrightarrow x_2)) \\ & = 2g \cdot g \int dx_1 dx_2 D_{\mu}^2(x_1 - x_2) D_{\mu}(y_1 - x_1) D_{\mu}(y_2 - x_1) D_{\mu}(y_3 - x_2) D_{\mu}(y_4 - x_2) \end{aligned} \quad (13)$$

where the \simeq indicates that the t and u channels have been omitted.

For n -th order there are n interaction points $x_i, i = 1 \dots n$. There are $n(n-1)$ possibilities to attach the external pairs (y_1, y_2) and (y_3, y_4) to interaction points, and there are $(n-2)!$

possibilities to join these two points by a closed path of $n - 1$ blobs $\frac{16\pi^2}{2}D_\mu^2(x_i - x_j)$. E.g. in fourth order, after fixing an attachment of the external legs, there remain two possibilities:

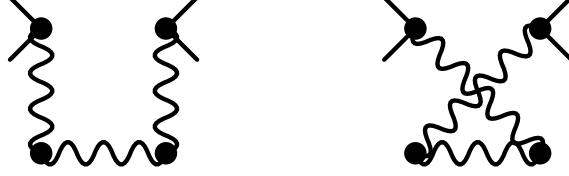


Fig. 4

This combinatorial factor $n(n - 1)((n - 2)!)$ precisely cancels the $\frac{1}{n!}$ from the n -th order perturbation expansion.

So all vertices have coupling constant g , except the first one that has $2g$ (actually this factor 2 may be understood, in the language of conventional perturbation theory, as a final state symmetry factor $(y_3, y_4) + (y_4, y_3) \rightarrow 2(y_3, y_4)$).

Therefore, we indeed find a series like in Fig. 3 which, in momentum space and after amputation of the external legs, reads

$$P(q^2) := 2g(1 + gS(q^2) + (gS(q^2))^2 + \dots) = \frac{2g}{1 - gS(q^2)} \quad (14)$$

where q is the total incoming momentum and $S(q^2)$ is just the blob (without external legs)

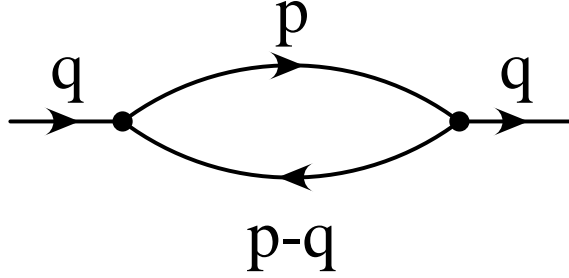


Fig. 5

and may be evaluated by standard methods:

$$\begin{aligned} S(q^2) &= \int \frac{d^2p}{(2\pi)^2} \frac{-1}{p^2 + \mu^2} \frac{-1}{(p - q)^2 + \mu^2} = \int \frac{d^2p}{(2\pi)^2} \int_0^1 \frac{dx}{[p^2 + 2pq(x - 1) + q^2(1 - x) + \mu^2]^2} \\ &= \frac{1}{4\pi} \int_0^1 \frac{dx}{q^2 x(1 - x) + \mu^2} = \frac{1}{4\pi(-q^2)} \int_0^1 \frac{dy}{y^2 + (\frac{4\mu^2}{-q^2} - 1)} \\ &= \frac{1}{4\pi(-q^2)} \frac{1}{R(q^2)} \arctan \frac{1}{R(q^2)}, \end{aligned} \quad (15)$$

$$R(q^2) := \sqrt{\frac{4\mu^2}{-q^2} - 1} \quad (16)$$

where we used the fact that, for the bound state, $-q^2$ has to be beyond the threshold, $-q^2 < 4\mu^2$.

Now we simply have to insert this result into (14) in order to get the mass pole:

$$1 = \frac{g}{4\pi(-q^2)} \frac{1}{R(q^2)} \arctan \frac{1}{R(q^2)}. \quad (17)$$

For small fermion masses both g and $R(q^2)$ are very small. Therefore, the leading order result will stem from a matching between these two factors, where we may set $\frac{1}{-q^2} = \frac{1}{4\mu^2}$ and $\arctan \frac{1}{R(q^2)} = \frac{\pi}{2}$. Doing so we get

$$R(q^2) = \frac{g}{32\mu^2} \quad (18)$$

or

$$-q^2 = 4\mu^2 \frac{1}{1 + (\frac{g}{32\mu^2})^2} \simeq 4\mu^2 (1 - \frac{\pi^4 m^2 \Sigma^2 \cos^2 \theta}{16\mu^4}) < 4\mu^2, \quad (19)$$

$$-q^2 = 4\mu^2 (1 - 0.4892 \frac{m^2}{\mu_0^2} \cos^2 \theta + o(\frac{m^3}{\mu_0^3})) \quad (20)$$

which is the bound state mass we are looking for. Observe that the pole mass equation (18) remains perfectly sensible in the limit $g \rightarrow 0$, where it leads to the *exact* result $-q^2 = 4\mu_0^2$ (no interaction). This shows a posteriori that our approximation in Fig. 3 and formula (14) indeed is justified for sufficiently small fermion mass.

We find from (20) that the leading bound state mass correction is already of second order in m . Therefore, in order to be able to express the result consistently in the zero order Schwinger mass μ_0 , one should include the Schwinger mass corrections up to second order. This corrections have been computed in [18] for an *external* boson. Because of the complicated exponential structure of the interaction term (see (10)) it is not obvious that the same corrections remain true for internal boson lines. More precisely, it is obvious that the types of vertices are the same, it is however not completely obvious that even the combinatorial factors remain the same.

So let us shortly check it by investigating the perturbation formula (10). A typical first order correction graph looks like

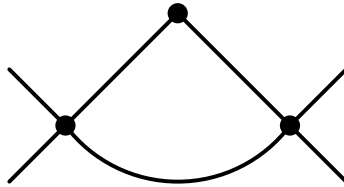


Fig. 6

and is of third order, $\frac{m^3}{3!}$. There are $3 \cdot 2 = 6$ possible attachments of the external pairs. From the exponent $\exp(\pm 4\pi D_{\mu_0}(x_i - x_j))$ we need 3 first order coefficients instead of one second order one. Therefore we get, in a symbolic notation (indicating integrations by \times and amputating external lines, $D \equiv D_{\mu_0}(x)$)

$$2g((4\pi)^3 m^2 \Sigma^2 \cos^2 \theta D \cdot D \times D) = 2g(g \cdot 2D(4\pi m \Sigma \cos \theta D \times D)) =: 2g(g \cdot 2D\delta_1 D). \quad (21)$$

The second order mass correction stems from the fourth order perturbation expansion $\frac{m^4}{4!}$. There are $4 \cdot 3$ attachments, and for a fixed attachment each contributing type of term occurs twice. We again need three first order propagators $4\pi D_{\mu_0}$, and one complete exponential $E_{\pm}(x)$. For a fixed attachment the following graphs contribute:

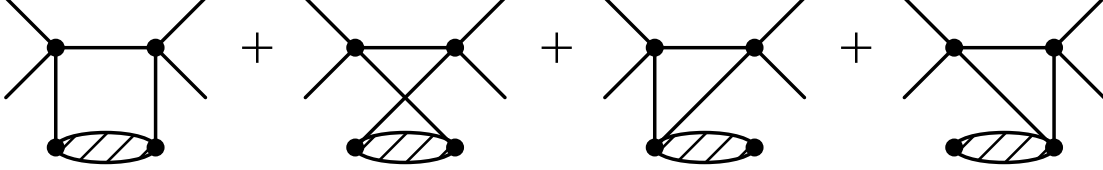


Fig. 7

and lead to

$$2gg \cdot 2D \left[4\pi m^2 \frac{\Sigma^2}{2} \left(\cos 2\theta (E_+ \cdot D \times D + E_+ \times D \times D) + E_- \cdot D \times D - E_- \times D \times D \right) \right] \\ =: 2g^2 \cdot 2D\delta_2 D. \quad (22)$$

Both $\delta_1 D$ and $\delta_2 D$ are precisely like for external bosons (see [18]), therefore the result (9) may be used and we find for the bound state mass

$$M_B^2 = 4\mu_0^2 \left(1 + 2e^\gamma \cos \theta \frac{m}{\mu_0} + e^{2\gamma} (A \cos 2\theta + B) \frac{m^2}{\mu_0^2} - \frac{\pi^2 e^{2\gamma}}{64} \cos^2 \theta \frac{m^2}{\mu_0^2} + o\left(\frac{m^3}{\mu_0^3}\right) \right) \\ = 4\mu_0^2 \left(1 + 3.5621 \cos \theta \frac{m}{\mu_0} + 5.2361 \frac{m^2}{\mu_0^2} - 2.3379 \cos 2\theta \frac{m^2}{\mu_0^2} + o\left(\frac{m^3}{\mu_0^3}\right) \right). \quad (23)$$

This is our final result.

4 Summary

We have reached our aim of computing the boson-boson bound state mass within (fermion) mass perturbation theory. The existence of this bound state is a necessary consequence of the attractive force between the Schwinger bosons in $1+1$ dimensions.

In the actual computation we used an additional approximation besides mass perturbation, namely we chose the lowest possible Fock state (two bosons) in any order of perturbation theory. This approximation could be shown to lead to a summation of graphs like in Φ^4 -theory. The approximation is justified for small fermion mass, because there this lowest Fock state is near its threshold singularity, or, differently stated, because it is this and only this Fock state that survives the limit of vanishing fermion mass.

Because of the approximation the bound state pole mass remained polynomial in the fermion mass even after the summation of all contributing graphs. Actually it turned out quadratically in the fermion mass in leading order.

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